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The Waring formula and fusion rings

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Abstract

The potential that generates the cohomology ring of the Grassmannian is given in terms of the elementary symmetric functions using the Waring formula that computes the power sum of roots of an algebraic equation in terms of its coefficients. As a consequence, the fusion potential for $su(N)_K$ is obtained. This potential is the explicit Chebyshev polynomial in several variables of the first kind. We also derive the fusion potential for $sp(N)_K$ from a reciprocal algebraic equation. This potential is identified with another Chebyshev polynomial in several variables. We display a connection between these fusion potentials and generalized Fibonacci and Lucas numbers. In the case of $su(N)_K$ the generating function for the generalized Fibonacci numbers obtained are in agreement with Lascoux using the theory of symmetric functions. For $sp(N)_K$, however, the generalized Fibonacci numbers are obtained from new sequences.

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1. Introduction

In the work of Gepner [1], the fusion potential for $su(N)_K$ was obtained as a perturbation of the Landau–Ginzberg potential that generates the cohomology ring of the Grassmannian. This implies that the fusion ring for $su(N)_K$ and the cohomology ring of the Grassman-nian are connected. The connection of these two rings may be understood as follows: Lesieur

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[2] noticed that the rules of multiplying Schubert cycles [3], which are the generators of the homology ring of the Grassmannian, formally coincide with the rules for multiplying Schur functions [4]. On the other hand, the characters of the irreducible representation of $su(N)$ turn out to be given by the Schur functions [5] with some constraint which is exactly the perturbation mentioned above. Therefore, we learn that the product of characters is the same as a product of Schur functions with this constraint which, in turn, implies the connection between the cohomology ring for the Grassmannian and fusion ring for $su(N)_K$.

The potential that generates the cohomology ring of the Grassmannian turns out to be given by a power sum symmetric function in the Chern roots [6] that we identify with the roots of an algebraic equation, say of degree r, i.e., of the form

$$y^r + a_1 y^{r-1} + \cdots + a_{r-1} y + a_r = 0. \tag{1}$$

Geometrically, the degree r is the rank of the quotient bundle on the Grassmannian and the coefficients of the algebraic equation (elementary symmetric functions) correspond to the Chern classes of this bundle. With this interpretation in mind, the algebraic equation (1) is nothing but the definition of the Chern classes of a vector bundle of rank r given by Grothendieck [7], where y is identified with the fundamental class of degree 2 on the associated projective bundle.

In this paper, we use the Waring formula to express the power sum symmetric function in the Chern roots in terms of the elementary symmetric functions and hence obtain the cohomology potential for the Grassmannian and the fusion potentials for $su(N)_K$ and $sp(N)_K$. The algebraic equation from which the $su(N)_K$ fusion potential is obtained is the one for which $r = N$ and $a_N = 1$, whereas, for $sp(N)_K$, it turns out to be a reciprocal algebraic equation [8] of order $2N$, with the last coefficient equal to 1 and $a_{2N-i} = a_i$.

In our formulation, the fusion potential written in terms of the elementary symmetric functions is the explicit generalization of the Chebyshev polynomial of one variable. Similarly, for the case of $sp(N)_K$, we obtain another Chebyshev polynomial in several variables. The one-variable Chebyshev polynomials of the first kind and second kind are known to be related to the ordinary Lucas numbers and Fibonacci numbers, respectively. In this paper, we find a relation to the generalized Fibonacci and Lucas numbers for the cases studied here.

Our paper is organized as follows: Section 2 gives a brief account of the cohomology ring in order to recall some facts and fix the notation. Section 3 will be devoted to the cohomology ring potential and its connection with the fusion ring for $su(N)_K$. The connection of the later with the generalized Chebyshev polynomial and the numbers of Fibonacci and Lucas will also be discussed. In Section 4, we will consider the $sp(N)_K$ fusion potential and its connection with the reciprocal algebraic equation. Here, we will find that the Chebyshev polynomial associated with $sp(N)_K$ is different from the one for $su(N)_K$ for $N \neq 1$. In this case, the Fibonacci and Lucas numbers are of degree $2N$. Our conclusions are outlined in Section 5.

2. The cohomology ring

In this section, we will recall briefly the definition of the cohomology ring of the Grassmannian [9] and the corresponding Landau–Ginzburg formulation [1,6] in order to fix our notation. The complex Grassmannian manifold here denoted by $G_r(C^n)$ is the space of r-planes in C^n, its cohomology ring denoted by $H^*(G_r(C^n))$ is a truncated polynomial ring in several variables given by

$$H^*(G_r(C^n)) \cong \frac{C[x_1, \dots, x_r, y_1, \dots, y_{n-r}]}{I}, \tag{2}$$

where $x_i = c_i(Q)$ (for $1 \le i \le r$) are the Chern classes of the quotient bundle Q of rank r, i.e., $x_i \in H^{2i}(G_r(C^n))$ and $y_j = c_j(S)$ (for $1 \le j \le n-r$) are the Chern classes of the universal bundle S of rank $n-r$. The ideal I in $C[x_1, \dots, x_r, y_1, \dots, y_{n-r}]$ is given by

$$(1 + x_1 + x_2 + \cdots + x_r)(1 + y_1 + y_2 + \cdots + y_{n-r}) = 1, \tag{3}$$

which is the consequence of the tautological sequence on $G_r(C^n)$:

$$0 \to S \to V \to Q \to 0,$$

where $V = G_r(C^n) \times C^n$. By using Eq. (3), one may rewrite $H^*(G_r(C^n))$ as

$$H^*(G(C^n)) \cong \frac{C[x_1, \dots, x_r]}{y_j}, \tag{4}$$

where y_j are expressed in terms of x_i, and $y_j = 0$ for $n-r+1 \le j \le n$, and $x_0 = y_0 = 1$. The classes y_j can be written inductively as a function of $x_1, \dots, x_r$ via

$$y_j = -x_1 y_{j-1} - \cdots - x_{j-1} y_1 - x_j \quad \text{for} \quad j = 1, \dots, n-r. \tag{5}$$

We will give later on an explicit formula for the y_j's in terms of the x_i's without the use of induction.

In the Landau–Ginzburg formulation, the potential that generates the cohomology ring of the Grassmannian as explained in [1,6,10] is given by

$$W_{n+1}(x_1, \dots, x_r) = \sum_{i=1}^{r} \frac{q_i^{n+1}}{n+1}, \tag{6}$$

where x_i and q_i are related by

$$x_i = \sum_{1 \le l_1 < l_2 < \cdots < l_i \le r} ql_1 ql_2 \cdots ql_i. \tag{7}$$

Usually the description of the cohomology ring is given in terms of the q_i variables, however, in the next section, we will write down the potential in terms of the x_i's, i.e., as a solution to the above system of equations. Note that, as was shown explicitly in [10], the cohomology ring of the Grassmannian is given by

$$\frac{\partial W_{n+1}}{\partial x_i} = (-1)^n y_{n+1-i} \quad \text{for} \quad 1 \le i \le r, \tag{8}$$

implying that $\mathrm{d}_i W_{n+1} = 0$ for $i = 1, \dots, r$.

3. The cohomology ring potential

A formula for the Landau–Ginzburg potential $W_{n+1}(x_1,\dots,x_r)$ is given in terms of the generators of $H^*(G_r(C^n))$, and when we consider the potential $W_n(x_1,\dots,x_r)$ instead with $n=N+k$, $r=N$ and $x_N=1$ we obtain the fusion potential of the $su(N)_K$ [1]. The fusion potential in this formulation is the explicit generalized Chebyshev polynomial in several variables. The ordinary Fibonacci and the Lucas numbers are known to be connected to Chebyshev polynomial of the second kind and the first kind, respectively [13]. Here we will find the connection between the fusion potential of the $su(N)_K$ algebra and the kth order Fibonacci and the Lucas numbers. The following formulae for the potential, the classes y_j in terms of the x_i classes and, in general, the connection between Segre classes of any vector bundle of rank n in terms of Chern classes are first proposed, and then later proved using the theory of symmetric functions [11].

Proposition 1. *The potential $W_{n+1}(x_1,\dots,x_r)$ that generates the cohomology ring of the Grassmannian $H^*(G_r(C^n))$ in terms of the generators $x_i=c_i(Q)$ for $1\le i\le r$ is given by the formula*

$$W_{n+1}(x_1,\dots,x_r)=\sum_{k_1=0}^{[(n+1)/2]}\cdots\sum_{k_{r-1}=0}^{[(n+1)/r]}\frac{(-1)^{k_1+2k_2+\cdots+(r-1)k_{r-1}}}{k_1!\cdots k_{r-1}!}\times\frac{(n-\sum_{j=1}^{r-1}jk_j)!}{(n+1-\sum_{j=2}^{r}jk_{j-1})!}x_1^{n+1-2k_1-\cdots-rk_{r-1}}x_2^{k_1}\cdots x_r^{k_{r-1}}. \tag{9}$$

The above formula reduces to the fusion potential of $su(N)_K$ algebra when we consider the potential $W_n(x_1,\dots,x_r)$ instead, with $n=N+k$, $r=N$ and $x_N=1$ which in turn is the explicit multidimensional analogue of Chebyshev polynomial of the first kind. Finally the fusion potential and the multidimensional analogue of the Chebyshev polynomial of the second kind are shown to be related the kth order Lucas and Fibonacci numbers, respectively.

To prove the above formula, we use the fundamental theorem on symmetric functions [4], which states that any symmetric function can be written as a polynomial in the elementary symmetric functions. The potential for the cohomology ring of the Grassmannian, $H^*(G_r(C^n))$ is generated by

$$W_{n+1}(x_1,\dots,x_r)=\sum_{i=1}^{r}\frac{q_i^{n+1}}{n+1},$$

i.e., the power sum symmetric functions in the Chern roots, q_i.[1] From [11], we learn that there is an explicit formula for the power sum in terms of the elementary symmetric functions. As a matter of fact, this formula was given by Waring [8,12] in connection with

[1] q_i are the formal variables satisfying $\sum_{i=0}^{r}x_it^i=\prod_{i=1}^{r}(1+q_it)$.

the theory of algebraic equations in which he found a general expression for the power sum of the roots of an algebraic equation of order r in terms of its coefficients. This formula reads as

$$s_n = n \sum (-1)^{n+l_1+\cdots+l_r} \frac{(l_1+\cdots+l_r-1)!}{l_1!\cdots l_r!} x_1^{l_1}\cdots x_r^{l_r}, \tag{10}$$

where s_n denotes the power sum, the x_i's are the elementary symmetric functions, and the summation is taken over all positive integers or zero such that $l_1+2l_2+\cdots+rl_r=n$.

It is clear from Eq. (10) that we obtain the formula for the cohomology potential given by Eq. (9): simply shift n to $n+1$ in Eq. (10), set $l_1 = n+1-2l_2-\cdots-rl_r$ and now by making the change of variables $l_2 = k_1, \ldots, l_r = k_{r-1}$, the formula is obtained. To prove that the potential $W_{n+1}(x_1,\ldots,x_r)$ generates the cohomology ring, we need the following formula that relates the Chern classes of the universal bundle, y_j, to the Chern classes of the quotient bundle x_i

$$y_j = (-1)^j \sum_{k_1=0}^{[j/2]} \cdots \sum_{k_{r-1}=0}^{[j/r]} \frac{(-1)^{k_1+2k_2+\cdots+(r-1)k_{r-1}}}{k_1!\cdots k_{r-1}!} \times \frac{(j-\sum_{l=1}^{r-1} lk_l)!}{(j-\sum_{l=2}^{r} lk_{l-1})!} x_1^{j-2k_1-\cdots-rk_{r-1}} x_2^{k_1}\cdots x_r^{k_{r-1}}. \tag{11}$$

Again we can use the theory of symmetric functions to prove the equation. This time however, we use the relation between the homogeneous product sum (also called the completely symmetric functions) and the elementary symmetric functions. The Segre classes, denoted by s_j, of a vector bundle of rank n has an expression similar to that for the Chern classes of the universal bundle but with $r=n$ and with j allowed to take the values $1, 2, \ldots, n$. Now the proof that $W_{n+1}(x_1,\ldots,x_r)$ generates the cohomology ring follows by differentiating this potential with respect to x_i, $1 \le i \le r$. Thus, we obtain

$$\frac{\partial W_{n+1}}{\partial x_i} = (-1)^{i-1} \sum_{k_1=0}^{[(n+1-i)/2]} \cdots \sum_{k_{r-1}=0}^{[(n+1-i)/r]} \frac{(-1)^{k_1+2k_2+\cdots+(r-1)k_{r-1}}}{k_1!\cdots k_{r-1}!} \times \frac{(n+1-i-\sum_{j=1}^{r-1} jk_j)!}{(n+1-i-\sum_{j=2}^{r} jk_{j-1})!} x_1^{n+1-i-2k_1-\cdots-rk_{r-1}} x_2^{k_1}\cdots x_r^{k_{r-1}}. \tag{12}$$

From Eq. (11), we see that y_{n+1-i} is exactly the expression for $\partial W_{n+1}/\partial x_i$ up to $(-1)^n$ which is zero for $i=1,\ldots,r$, by definition of the cohomology ring. Therefore, $\partial W_{n+1}/\partial x_i = (-1)^n y_{n+1-i}$ implying that $\mathrm{d}_i W = 0$ for $i = 1,\ldots,r$. This shows the isomorphism between the usual definition of the cohomology ring of the Grassmannian $H^*(G_r(C^n))$ and the Landau–Ginzburg formulation.

Now, we come to the connection between the cohomology potential and fusion potential of $su(N)_K$ algebra. We consider the potential $W_n(x_1,\ldots,x_r)$ with $n = N+k$, $r = N$ and set $x_N = 1$ in the expression of $W_n(x_1, \ldots, x_r)$ to obtain the following potential:

$$\begin{aligned} &W_{N+K}(x_1, \ldots, x_N = 1) \\ &= \sum_{k_1=0}^{[(N+K)/2]} \cdots \sum_{k_{N-1}=0}^{[(N+K)/N]} \frac{(-1)^{k_1+2k_2+\cdots+(N-1)k_{N-1}}}{k_1! \cdots k_{N-1}!} \\ &\quad \times \frac{(N+K-1-\sum_{j=1}^{N-1} jk_j)!}{(N+K-\sum_{j=2}^{N} jk_{j-1})!} x_1^{N+K-2k_1-\cdots-Nk_{N-1}} x_2^{k_1} \cdots x_{N-1}^{k_{N-2}}, \end{aligned} \tag{13}$$

this potential is no longer quasihomogeneous. The quasihomogeneous part of this potential is obtained by setting $k_{N-1} = 0$. To see that this potential is the natural analogue of Chebyshev polynomial of the first kind in several variables, we specialize the potential to the case of $su(2)_K$ and find

$$(2+K)W_{2+K}(x) = (2+K) \sum_{l=0}^{[(K+2)/2]} \frac{(-1)^l}{l!} \times \frac{(k+1-l)!}{(k+2-2l)!} x^{K+2-2l}. \tag{14}$$

By setting $n = K + 2$, one has

$$nW_n(x) = n \sum_{l=0}^{[n/2]} \frac{(-1)^l}{l!} \times \frac{(n-1-l)!}{(n-2l)!} x^{n-2l}. \tag{15}$$

This is exactly the Chebyshev polynomial of the first kind [13]. In this representation the Chebyshev polynomial is monic and with integer coefficients.

It remains to be seen that the analogue of the Chebyshev polynomial of the first kind in several variables is the fusion ring of the $su(N)_K$ algebra. This is a simple consequence of the relation between our cohomology potential and the Chern classes of the universal bundle S. By using Eq. (12) in the $su(N)_K$ case, one has

$$y_{N+K-i} = (-1)^{i+1} \frac{\partial W_{N+K-i}}{\partial x_i} \quad \text{for } 1 \le i \le N-1, \tag{16}$$

which is the ideal of the fusion ring for $su(N)_K$ [1]. Therefore the fusion ring for $su(N)_K$ is $R = C[x_1, \ldots, x_{N-1}]/(y_{K+1}, y_{K+2}, \ldots, y_{K+N-1})$. In terms of Young tableaux, this is equivalent to setting to zero all reduced tableaux (no columns with N boxes) for which the first row has length equal to $K + 1$, this is the level truncation. When the Giambeli-like formula [1] is applied to the completely symmetric representation, the fusion ideal for $su(N)_K$ reads

$$\underbrace{[1, \ldots, 1]}_{j} = \det x_{1+l-s} \quad \text{for } 1 \le l, s \le j,\ K+1 \le j \le N+K-1. \tag{17}$$

Therefore the completely symmetric function y_j given by (11) is the explicit expression for the Giambeli-like formula when restricted to $[1, \ldots, 1]$ (with j entries), where $K + 1 \le j \le N + K - 1$.

From [1] we learn that there are two ways to obtain the fusion potential for $su(N)_K$ algebra. One way is to use the following expression:

$$W_{N+K}(x_1, \dots, x_N = 1) = \frac{(-1)^{N+K}\, \mathrm{d}^{N+K}}{(N+K)!\, \mathrm{d}t^{N+K}} \log \left(\sum_{i=0}^{N} (-1)^i x_i t^i \right) \bigg|_{t=0} \tag{18}$$

with $x_0 = x_N = 1$. Alternatively, we use the recursion relation satisfied by the potential

$$\sum_{i=0}^{N} (-1)^i x_i (N+s-i) W_{N+s-i} = 0. \tag{19}$$

Therefore, our expression for the fusion potential is simpler and more transparent. It gives the integrability of the Chern classes y_j (completely symmetric functions) to a potential as a consequence of the cohomology of the Grassmannian. Furthermore, from our fusion potential which is the explicit Chebyshev polynomial of the first kind in several variables, one can read off directly the $su(N)_K$ fusion potential for any N and K.

Before we make the connection between the fusion potential of $su(N)_K$ algebra and the Fibonacci numbers and Lucas numbers of kth order, we will first give the definition of these numbers. We will then recall the connection between the ordinary Fibonacci and Lucas numbers with the Chebyshev polynomial of one variable.

Definition 1. The kth order Fibonacci numbers F_{n+1} and Lucas numbers L_n are defined, respectively, by $F_{n+1} = F_n + F_{n-1} + \cdots + F_{n-k}$, $L_n = L_{n-1} + L_{n-2} + \cdots + L_{n-k}$, with the initial conditions $F_{-k+1} = \cdots = F_{-1} = 0$ and similarly for the L_n's.

For $k = 2$, these are the definitions of the ordinary Fibonacci and Lucas numbers which are given by $F_{n+1} = F_n + F_{n-1}$ and $L_n = L_{n-1} + L_{n-2}$, i.e., any number is the sum of the previous two. The Chebyshev polynomial of the second kind $U(x/2)$ is known to be related to the ordinary Fibonacci numbers, and the Chebyshev polynomial of the first kind $T(x/2)$ is known to be related to the Lucas numbers [13] via the following specializations:

$$F_{n+1} = \frac{S_n(\mathrm{i})}{\mathrm{i}^n}, \quad n = 0, 1, \dots \quad (\mathrm{i}^2 = -1), \tag{20}$$

where

$$S_n(x) = U\left(\frac{x}{2}\right) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} x^{n-2k}, \tag{21}$$

and

$$L_n = \frac{C_n(\mathrm{i})}{\mathrm{i}^n}, \quad n = 0, 1, \dots, \tag{22}$$

where

$$C_n(x) = 2T\left(\frac{x}{2}\right) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}. \tag{23}$$

By applying a similar procedure, the analogue of the Chebyshev polynomial of the second kind in several variables and the fusion potential reduce to the following two sequences of numbers, respectively:

$$F_{n+1} = \sum_{l_1=0}^{[n/2]} \cdots \sum_{l_{k-1}=0}^{[n/k]} \frac{1}{l_1! \cdots l_{k-1}!} \frac{(n - \sum_{j=1}^{k-1} j l_j)!}{(n - \sum_{j=2}^{k} j l_{j-1})!}, \tag{24}$$

$$L_n = n \sum_{l_1=0}^{[n/2]} \cdots \sum_{l_{k-1}=0}^{[n/k]} \frac{1}{l_1! \cdots l_{k-1}!} \frac{(n - 1 - \sum_{j=1}^{k-1} j l_j)!}{(n - \sum_{j=2}^{k} j l_{j-1})!}. \tag{25}$$

One can see that these numbers are indeed those given in the definition above, for example, for $k = 2$ they are the ordinary Fibonacci and Lucas numbers, respectively. For $k = 3$ we have the third order Fibonacci and Lucas numbers and we have computed the first few of them as given below

$$F_{n+1}^{(3)} = 1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots, \tag{26}$$

$$L_n^{(3)} = 1, 3, 7, 11, 21, 39, 71, 131, \ldots. \tag{27}$$

In terms of the level K, and for a fixed value of N in the $su(N)_K$, one can see that the first term in the Fibonacci sequence will start at $n = N + K + 1$, whereas that of the Lucas sequence will start at $n = N + K$, and N is identified with the order of these two series.

The formula given above corresponding to kth order Fibonacci numbers is in full agreement with that obtained by Lascoux [14] in which he showed by using the theory of symmetric functions that kth order Fibonacci numbers are given by the following multinomial

$$F_{n+1} = \sum_I \binom{\ell(I)}{m_1, \ldots, m_k}, \tag{28}$$

where the summation is taken over all partitions $I = 1^{m_1} 2^{m_2} \cdots$ of weight $n = m_1 + 2m_2 + \cdots + km_k$ and $\ell(I)$ is the length of the partition $m_1 + m_2 + \cdots + m_k$.

The equivalence of our formula for the kth order Fibonacci numbers and those given by Lascoux follows by expanding the multinomial (28), and fixing m_1 as $m_1 = n - 2m_2 - \cdots - km_k$ and then changing the variables as we did before to obtain the cohomology potential.

Although the expression for the kth order Lucas number were not given in [14], we can see however that the equivalent formula for these numbers is

$$L_n = n \sum_I \binom{\ell(I) - 1}{m_1, \ \ldots, m_k}. \tag{29}$$

4. $sp(N)_K$ fusion potential and the reciprocal algebraic equation

We recall from the last section that the Waring formula computes the power sum of roots of an algebraic equation in terms of its coefficients. These coefficients are identified with the elementary symmetric functions in terms of the Chern roots, they are given by $x_i = \sum_{l \leq l_1,\dots,l_i \leq r} q_{l_1} q_{l_2} \cdots q_{l_i}$. The algebraic equation from which one computes the cohomology ring potential has the form

$$y^r + a_1 y^{r-1} + \cdots + a_{r-1} y + a_r = 0, \tag{30}$$

where the coefficients a_i are identified with the elementary symmetric functions, and the roots of this algebraic equation are q_i for $i = 1, \dots, r$.

The fusion potential for the $su(N)_K$ algebra may be obtained from the following algebraic equation:

$$y^N + a_1 y^{N-1} + \cdots + a_{N-1} y + 1 = 0. \tag{31}$$

The last coefficient is set equal to 1 due to the constraints $x_N = q_1 q_2 \dots q_N = 1$, which in turn corresponds to the fact that the determinant of the maximal torus of $SU(N)$ group is the identity. The diagonal elements of this torus are $q_i = \mathrm{e}^{i(\theta_i - \theta_{i-1})}$ for $i = 1, \dots, N$ with the convention $\theta_0 = \theta_N = 1$. With this motivation in mind, one would like to know whether one can write down an algebraic equation corresponding to groups other than $SU(N)$, in particular, the unitary symplectic group $Sp(N)$. Having written down such an algebraic equation, the fusion potential for the $sp(N)_K$ algebra is obtained using the Waring formula. This turns out to be true as we will shortly see.

From [15] we learn that any $n \times n$ unitary symplectic matrix (with $n = 2m$) can be diagonalized with diagonal elements of the form q_i and q_i^{-1} for $i = 1, \dots, m$, and with determinant equal to 1. Therefore the algebraic equation that we are looking for is the one for which both q_i and q_i^{-1} are roots and where the last coefficient is equal to 1. Such algebraic equations are called reciprocal equations of the first class [8]. In our case, this algebraic equation has the form

$$y^{2m} + a_1(y^{2m-1} + y) + a_2(y^{2m-2} + y^2) + \cdots + a_m y^m + 1 = 0, \tag{32}$$

where $a_i = a_{2m-i}$. Note that in this case, the elementary symmetric functions are functions of both q_i and q_i^{-1} that we denote by E_i. Now, the natural power sum to consider for the reciprocal algebraic equation has the form

$$W_n(E_1, \dots, E_m) = \frac{1}{n} \sum_{i=1}^{m} (q_i^n + q_i^{-n})$$

as both q_i and q_i^{-1} are roots of Eq. (32). This is exactly the form proposed by Gepner and Schwimmer [16]. Therefore, by applying the Waring formula to this expression, one obtains

$$\begin{aligned} W_n(E_1, \dots, E_m) = {} & \sum_{k_1=0}^{[n/2]} \cdots \sum_{k_{2m-1}=0}^{[n/2m]} \frac{(-1)^{k_1 + 2k_2 + \cdots + (2m-1)k_{2m-1}}}{k_1! \cdots k_{2m-1}!} \\ & \times \frac{(n - 1 - \sum_{l=1}^{2m-1} l k_l)!}{(n - \sum_{l=2}^{2m} l k_{l-1})!} E_1^{g(n,m)} E_2^{k_1 + k_{2m-1}} \cdots E_m^{k_{m-1}}, \end{aligned} \tag{33}$$

where $g(n,m) = n - 2k_1 - \cdots - (2m-2)k_{2m-2} - 2mk_{2m-1}$. In obtaining the above equation, we have used the condition $l_1 + 2l_2 + \cdots + 2ml_{2m} = n$, and the change of variables $l_2 = k_1, \ldots, l_{2m} = k_{2m-1}$.

In the following we will briefly recall the classical tensor ring for $sp(N)$ and the modified fusion ring [17,18], namely, the $sp(N)_K$ algebra and hence write explicitly the fusion potential for the latter. The classical tensor ring for $sp(N)$ is the finite ring, $R = C[\chi_1, \ldots, \chi_N]/I_\mathrm{C}$, where χ_j are the characters of the fundamental representation corresponding to a single column of length j. These characters are related to the elementary symmetric function E_j [19] by

$$\chi_j = E_j - E_{j-2}. \tag{34}$$

The classical ideal I_C is obtained by using this equation with the property that $E_j = E_{2N-j}$ and $E_0 = E_{2N} = 1$ [2], which follows from its generating function [19]:

$$E(t) = \sum_{j=0}^{\infty} E_j t^j = \prod_{i=1}^{N} (1 + q_i t)(1 + q_i^{-1} t). \tag{35}$$

Therefore, the classical ideal I_C is given by

$$\chi_{N+1} = 0, \qquad \chi_{N+2} + \chi_N = 0, \ldots, \qquad \chi_j + \chi_{2N+2-j} = 0.$$

The $sp(N)_K$ fusion ring is obtained by a further modification of the classical $sp(N)$ tensor ring in which tableaux with more than K columns are eliminated. This is equivalent to writing the $sp(N)_K$-fusion ring as $R = C[\chi_1, \ldots, \chi_N]/I_\mathrm{f}$, where the ideal I_f is given by

$$J_{K+1} = 0, \qquad J_{K+2} + J_K = 0, \ldots, \qquad J_{K+N} + J_{K-N+2} = 0. \tag{36}$$

J_j represents the character of the single row tableaux of length j which is a completely symmetric function whose generating function is [19]

$$J(t) = \sum_{j=0}^{\infty} J_j t^j = \prod_{i=1}^{N} \frac{1}{(1 - q_i t)(1 - q_i^{-1} t)}. \tag{37}$$

Since $E(t)J(-t) = 1$, the completely symmetric functions can be written in terms of the elementary symmetric functions as will be given explicitly below.

The truncation given by Eq. (36) can be written as the ideal generated by setting to zero the derivative of the potential W_n (33) for certain values of n and m. This means that the fusion ring for $sp(N)_K$ can be written as $R = C[\chi_1, \ldots, \chi_N]/\mathrm{d}W_n$. To see this we differentiate the potential W_n with respect to E_i finding:

$$\frac{\partial W_n}{\partial E_i} = \begin{cases} (-1)^{i+1}(J_{n-i} + J_{n+i-2m}) & \text{for } 1 \le i \le n-1, \\ (-1)^{i+1} J_{n-i} & \text{for } i = m, \end{cases} \tag{38}$$

[2] This is exactly the condition that follows from a reciprocal algebraic equation of degree $2N$, with the last coefficient $a_{2N} = 1$.

where J_j is given explicitly by

$$J_j = (-1)^j \sum_{k_1=0}^{[j/2]} \cdots \sum_{k_{2m-1}=0}^{[j/2m]} \frac{(-1)^{k_1+2k_2+\cdots+(2m-1)k_{2m-1}}}{k_1! \cdots k_{2m-1}!} \times \frac{(j-\sum_{l=1}^{2m-1} lk_l)!}{(j-\sum_{l=2}^{2m} lk_{l-1})!} E_1^{j-2k_1-\cdots-(2m-2)k_{2m-2}-2mk_{2m-1}} E_2^{k_1+k_{2m-3}} \cdots E_m^{k_{m-1}}. \quad (39)$$

From Eq. (38), we see that the critical points of the potential W_n, $\partial W_n/\partial E_i = 0$ do indeed correspond to the fusion ideal $I_{\rm f}$ provided $n = N + K + 1$ and $m = N$.

The fusion potential for $sp(N)_K$ algebra is obtained by using the relation $\chi_j = E_j - E_{j-2}$. Setting $\chi_1 = x$ and $\chi_2 = y$, the fusion potentials for $sp(1)_K$ and $sp(2)_K$ are

$$(2+K)W_{2+K}(x) = (2+K) \sum_{l=0}^{[(K+2)/2]} \frac{(-1)^l}{l!} \frac{(K+1-l)!}{(K+2-2l)!} x^{K+2-2l}, \quad (40)$$

and

$$(3+K)W_{3+K}(x) = (3+K) \sum_{k_1=0}^{[(K+3)/2]} \sum_{k_2=0}^{[(K+3)/3]} \sum_{k_3=0}^{[(K+3)/4]} \frac{(-1)^{k_1+2k_2+3k_3}}{k_1!k_2!k_3!} \times \frac{(2+K-k_1-2k_2-3k_3)!}{(3+K-2k_1-3k_2-4k_3)!} x^{3+K-2k_1-2k_2-4k_3}(1+y)^{k_1}. \quad (41)$$

From Eq. (40) we see that this is the Chebyshev polynomial of the first kind for $su(2)_K$ as it should be, since $sp(1) = su(2)$. For levels $K = 1$ and $K = 2$, Eq. (41) gives the following potentials: $4W_4 = x^4 - 4x^2y + 2y^2 + 4y - 2$ and $5W_5 = x^5 - 5x^3y + 5xy^2 + 5xy - 5x$. These were the potentials obtained in [16] using the recursion relations.

The generalized Chebyshev polynomial in several variables for $sp(N)_K$ is obtained from the power sum W_n given in Eq. (33) with $n = N + K + 1$ and $m = N$,

$$W(E_1, \ldots, E_N) = \sum_{k_1=0}^{[(N+K+1)/2]} \cdots \sum_{k_{2N-1}=0}^{[(N+K+1)/2N]} \frac{(-1)^{k_1+2k_2+\cdots+(2N-1)k_{2N-1}}}{k_1! \cdots k_{2N-1}!} \times \frac{(N+K-\sum_{l=1}^{2N-1} lk_l)!}{(N+K+1-\sum_{l=2}^{2N} lk_{l-1})!} E_1^{f(N,K)} E_2^{k_1+k_{2N-1}} \cdots E_N^{k_{N-1}}, \quad (42)$$

where $f(N, K) = N + K + 1 - 2k_1 - \cdots - (2N-2)k_{2N-2} - 2Nk_{2N-1}$. This is different from the generalized Chebyshev polynomial in several variables for $su(N)_K$ for $N \neq 1$. The above results can be stated by the following proposition.

Proposition 2. *The fusion potential for the $sp(N)_K$ algebra is obtained from the power sum of the roots of a reciprocal algebraic equation of degree $2N$, the last coefficient of* *which is* 1. *The associated generalized Chebyshev polynomial of the first kind in terms of the elementary symmetric functions is given by Eq.* (42).

From the expression for the Chebyshev polynomial in several variables (42), one notes that the Lucas numbers are of order $2N$. Therefore these sequences are the same as those associated with $su(N)_K$ with N even. The difference however is that in the latter the sequence starts at $n = 2N + K$, whereas that associated with $sp(N)_K$ starts at $n = N + K + 1$. The point that is interesting to note is that the Fibonacci numbers associated to $sp(N)_K$ are combinations of two Fibonacci numbers. One can see this from (38) by considering the analogue of the Chebyshev polynomial that is associated with $sp(N)_K$ of the second kind. These numbers follow from $\partial W_{N+K+1}/\partial E_1 = J_{K+N} + J_{K-N+2}$, $N \neq 1$, to give the following Fibonacci type sequences of order $2N$.

$$\tilde{F}_{N+K+2} = F_{N+K+1} - F_{K-N+3}, \tag{43}$$

where

$$F_j = \sum_{k_1=0}^{[j/2]} \cdots \sum_{k_{2N-1}=0}^{[j/2N]} \frac{1}{k_1! \cdots k_{2N-1}!} \frac{(j - \sum_{l=1}^{2N-1} lk_l)!}{(j - \sum_{l=2}^{2N} lk_{l-1})!}. \tag{44}$$

As an example we have computed the Fibonacci numbers associated with $sp(2)_K$ using Eq. (43). The first few numbers are given below:

$$3, 5, 10, 19, 37, 71, 137, \ldots, \tag{45}$$

where the first term in this sequence corresponds formally to $K = 0$. We see that this sequence is a fourth order sequence and is different from the one obtained for $su(4)_K$. In fact the sequence given by Eq. (45) is a new sequence [20] and therefore the higher order sequences form new sequences given by Eq. (43).

5. Conclusions

In this paper we have seen that the cohomology potential that generates the cohomology ring of the Grassmannian $G_r(C^n)$, the fusion potentials for $su(N)_K$ and that for $sp(N)_K$ are obtained from suitable algebraic equations using the Waring formula (10) that computes the power sum of roots in terms of the elementary symmetric functions. The roots of these algebraic equations are in one-to-one correspondence with the elements of the diagonalized form of the unitary matrix groups $U(N)$, $SU(N)$ and $Sp(N)$.

In this algebraic formulation we see clearly that the isomorphism of Lie algebras should be translated into the identification of the corresponding algebraic equations. For example $su(2)$ and $sp(1)$ have the same algebraic equations which follow from Eqs. (31) and (32). As $sp(2) = so(5)$ the fusion potential for $so(5)_K$ should be given by Eq. (41) and hence the corresponding algebraic equation is

$$y^4 + \chi_1(y^3 + y) + (\chi_2 + 1)y^2 + 1 = 0, \tag{46}$$

indeed this is the case [21], where χ_1 is replaced by the spin character $B = (q_1^{1/2} + q_1^{-1/2})(q_2^{1/2} + q_2^{-1/2})$ and χ_2 is replaced by $e_1 = q_1 + q_1^{-1} + q_2 + q_2^{-1} + 1$. Therefore algebraic equations could be used for classifying fusion rings.

In this paper we also obtained an explicit connection between the fusion potentials and the Chebyshev polynomial in several variables for $su(N)_K$, which would be difficult to see in the formulation of [1,16]. These polynomials were shown to be related to the Fibonacci and the Lucas numbers. In the case of $sp(N)_K$ the Fibonacci numbers are of order $2N$ and appear to be new as they are different from those of $su(N)_K$ of the same order, however the Lucas numbers in both cases have the same order and belong to the same sequence.

We will see in our paper [22] where we showed that the ordinary Fibonacci numbers arise as intersections numbers on the moduli space of the holomorphic map from the sphere CP^1 into the Grassmannian $G_2(C^5)$.

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